

ON 2-ADIC DEFORMATIONS

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ABSTRACT. We compute the versal deformation ring of a split generic 2-dimensional representation $\chi_1 \oplus \chi_2$ of the absolute Galois group of \mathbb{Q}_p . As an application, we show that the Breuil–Mézard conjecture for both non-split extensions of χ_1 by χ_2 and χ_2 by χ_1 implies the Breuil–Mézard conjecture for $\chi_1 \oplus \chi_2$. The result is new for $p = 2$, the proof works for all primes.

1. INTRODUCTION

Let L be a finite extension of \mathbb{Q}_p with the ring of integers \mathcal{O} and residue field k . Let $\mathcal{G}_{\mathbb{Q}_p}$ be the absolute Galois group of \mathbb{Q}_p and let $\chi_1, \chi_2 : \mathcal{G}_{\mathbb{Q}_p} \rightarrow k^\times$ be continuous group homomorphisms, such that $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$, where ω is the cyclotomic character modulo p . We let

$$(1) \quad \rho_1 := \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}, \quad \rho_2 := \begin{pmatrix} \chi_1 & 0 \\ * & \chi_2 \end{pmatrix},$$

be non-split extensions. The assumption $\chi_1 \chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$ implies that both groups $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2)$ and $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_1)$ are 1-dimensional, thus ρ_1 and ρ_2 are uniquely determined up to an isomorphism.

Let \mathfrak{A} be the category of local artinian augmented \mathcal{O} -algebras with residue field k . Let D_1, D_2 be functors from \mathfrak{A} to the category of sets, such that for $A \in \mathfrak{A}$ and $i = 1, 2$, $D_i(A)$ is the set of deformations of ρ_i to A . Since ρ_1 and ρ_2 have scalar endomorphisms the functors D_1, D_2 are pro-represented by the universal deformation rings R_1, R_2 , respectively. We let $\rho_1^{\text{univ}}, \rho_2^{\text{univ}}$ be the universal deformations of ρ_1 and ρ_2 , respectively.

The representations $\rho_1^{\text{univ}}, \rho_2^{\text{univ}}$ are naturally pseudo-compact modules over the completed group algebra $\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]$. The main purpose of this note is to compute the ring $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$. The motivation for this computation stems from a companion paper [16], which shows that the category of pseudo-compact modules over this ring is naturally anti-equivalent to a certain subcategory of smooth $\text{GL}_2(\mathbb{Q}_p)$ -representations on \mathcal{O} -torsion modules. In fact, for this application we have to work with a fixed determinant, but we will ignore it in this introduction. If $p > 2$ then this result has been proved in [14, §B.2]. The proof there uses a result of Böckle [3], which realizes ρ_1^{univ} and ρ_2^{univ} concretely by writing down matrices for the topological generators. Böckle’s paper in turn uses results of Pink [17] on the classification of pro- p subgroups of $\text{SL}_2(R)$, where R is a p -adic ring with $p > 2$.

In this paper, we give a different argument, which works for all primes p , and obtain a more intrinsic description of $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$, which we will now

describe. Let $D^{\text{ps}} : \mathfrak{A} \rightarrow \text{Sets}$ be the functor, which sends A to the set of 2-dimensional A -valued determinants lifting the pair $(\chi_1 + \chi_2, \chi_1 \chi_2)$. The notion of an n -dimensional determinant has been introduced by Chenevier in [6]. If $p > n$ then it is equivalent to that of an n -dimensional pseudo-representation (pseudo-character). The functor D^{ps} is pro-represented by a complete local noetherian \mathcal{O} -algebra R^{ps} . We let $(t^{\text{univ}}, d^{\text{univ}})$ be the universal object. We show that $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$ is naturally isomorphic to the algebra opposite to the Cayley–Hamilton algebra $\text{CH}(R^{\text{ps}}) := R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]/J$, where J is the closed two-sided ideal in $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ generated by the elements $g^2 - t^{\text{univ}}(g)g + d^{\text{univ}}(g)$, for all $g \in \mathcal{G}_{\mathbb{Q}_p}$. We also show that $\text{CH}(R^{\text{ps}})$ is a free R^{ps} -module of rank 4 and compute the multiplication table for the generators, see Proposition 3.12. From this description we deduce that the centre of $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$ is naturally isomorphic to R^{ps} . As a part of the proof we show in Proposition 3.6 that mapping a representation to its trace and determinant induces isomorphisms $R^{\text{ps}} \xrightarrow{\cong} R_1$, $R^{\text{ps}} \xrightarrow{\cong} R_2$. In particular,

$$(t^{\text{univ}}, d^{\text{univ}}) = (\text{tr } \rho_1^{\text{univ}}, \det \rho_1^{\text{univ}}) = (\text{tr } \rho_2^{\text{univ}}, \det \rho_2^{\text{univ}}).$$

A key ingredient, in the most difficult $p = 2$ case, is the computation of $D^{\text{ps}}(k[\varepsilon])$ done in Proposition 3.4, where we follow very closely an argument of Bellaïche [1], and the description by Chenevier of R_1 and R_2 in [7]. In fact Chenevier has already shown that the maps are surjective, and become isomorphism after inverting 2.

In §5 we compute the versal deformation ring R^{ver} of $\chi_1 \oplus \chi_2$. We show that

$$R^{\text{ver}} \cong R^{\text{ps}}[[x, y]]/(xy - c),$$

where $c \in R^{\text{ps}}$ generates the reducibility ideal. This has been observed by Yongquan Hu and Fucheng Tan in [19], for $p > 2$, using results of Böckle, [3], which, as explained above, involves writing down matrices of “the most general form” for the topological generators. Our proof works as follows. If $\rho : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(k)$ is a continuous representation with semi-simplification isomorphic to $\chi_1 \oplus \chi_2$ then any lift of ρ to $A \in \mathfrak{A}$, $\rho_A : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(A)$ is naturally an $\text{CH}(R^{\text{ps}})$ -module. The idea is that if one understands the algebra $\text{CH}(R^{\text{ps}})$ well, one should be able just to write down the “most general” deformation of ρ . In view of structural results on Cayley–Hamilton algebras by Bellaïche–Chenevier [2, §1.4.3], we expect that this idea will be applicable in other contexts.

If $p = 2$ then using the description of R^{ver} above, we observe in Remark 5.3 that R^{ver} has two irreducible components which, via the map induced by taking determinants, correspond to the two irreducible components of the universal deformation ring of 1-dimensional representation $\chi_1 \chi_2$. This verifies a conjecture of Böckle and Juschka [4] in this case.

In §7 we show that the Breuil–Mézard conjecture formulated in [5], which describes the Hilbert–Samuel multiplicities of potentially semi-stable deformation rings, for ρ_1 and ρ_2 implies the Breuil–Mézard conjecture for the residual representation $\chi_1 \oplus \chi_2$. If $p > 2$ then the Breuil–Mézard conjecture in these cases has been proved by Kisin [11] as a part of his proof of Fontaine–Mazur conjecture. In [15], again under assumption $p > 2$, we have given a different local proof for the residual representations ρ_1 and ρ_2 . Yongquan Hu and Fucheng Tan observed in [19] that the Breuil–Mézard conjecture for ρ_1 and ρ_2 implies the result for $\chi_1 \oplus \chi_2$, thus obtaining a local proof also in the generic split case. They use results of Böckle to describe the versal deformation ring, and this forces them to assume $p > 2$. We

use our description of the versal ring, which works for all p , and closely follow their argument. The upshot is that in the companion paper [16] we apply the formalism developed in [15] to prove the Breuil–Mézard conjecture for ρ_1 and ρ_2 , when $p = 2$, and this paper implies the result in the split non-scalar case. We formulate our results in the language of cycles, as introduced by Emerton–Gee in [8].

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2. NOTATION

Let $D^{\text{ps}} : \mathfrak{A} \rightarrow \text{Sets}$ be a functor, which maps $(A, \mathfrak{m}_A) \in \mathfrak{A}$ to the set of pairs of functions $(t, d) : \mathcal{G}_{\mathbb{Q}_p} \rightarrow A$, such that the following hold: $d : \mathcal{G}_{\mathbb{Q}_p} \rightarrow A^\times$ is a continuous group homomorphism, congruent to $\chi_1 \chi_2$ modulo \mathfrak{m}_A , $t : \mathcal{G}_{\mathbb{Q}_p} \rightarrow A$ is a continuous function with $t(1) = 2$, and, which satisfy for all $g, h \in \mathcal{G}_{\mathbb{Q}_p}$:

- (i) $t(g) \equiv \chi_1(g) + \chi_2(g) \pmod{\mathfrak{m}_A}$;
- (ii) $t(gh) = t(hg)$;
- (iii) $d(g)t(g^{-1}h) - t(g)t(h) + t(gh) = 0$.

Such a pair $(t, d) : \mathcal{G}_{\mathbb{Q}_p} \rightarrow A$ corresponds to an A -valued 2-dimensional determinant in the sense of [6, Def. 1.15], see [6, Ex. 1.18]. Given such a pair, we let J be the closed two-sided ideal in $A[[\mathcal{G}_{\mathbb{Q}_p}]]$ generated by the elements $g^2 - t(g)g + d(g)$ for all $g \in \mathcal{G}_{\mathbb{Q}_p}$, and let

$$\text{CH}(A) := A[[\mathcal{G}_{\mathbb{Q}_p}]]/J$$

be the corresponding Cayley–Hamilton algebra.

Let H be the image and K be the kernel of the group homomorphism $\mathcal{G}_{\mathbb{Q}_p} \rightarrow k^\times \times k^\times$, $g \mapsto (\chi_1(g), \chi_2(g))$. Let P be the maximal pro- p quotient of K and let G be the quotient of $\mathcal{G}_{\mathbb{Q}_p}$ fitting into the exact sequence $1 \rightarrow P \rightarrow G \rightarrow H \rightarrow 1$. It follows from [14, Lem. A.1] that the map $A[[K]] \rightarrow \text{CH}(A)$ factors through $A[[P]] \rightarrow \text{CH}(A)$. In particular, $t(g)$ and $d(g)$ depend only on the image of g in G , and this induces an isomorphism of algebras $A[[G]]/J \cong \text{CH}(A)$, where J is the closed two-sided ideal of $A[[G]]$ generated by the elements $g^2 - t(g)g + d(g)$, for all $g \in G$. Since H is a finite group of order prime to p and P is pro- p , the surjection $G \twoheadrightarrow H$ has a splitting, so that $G \cong P \rtimes H$. We let

$$e_{\chi_1} := \frac{1}{|H|} \sum_{h \in H} [\chi_1](h)h^{-1}, \quad e_{\chi_2} := \frac{1}{|H|} \sum_{h \in H} [\chi_2](h)h^{-1},$$

where the square brackets denote the Teichmüller lifts to \mathcal{O} . We will denote by the same letters the images of these elements in $\text{CH}(A)$.

3. CAYLEY–HAMILTON ALGEBRAS

Lemma 3.1. *There is an isomorphism of $\mathcal{G}_{\mathbb{Q}_p}$ -representations:*

$$\text{CH}(k)e_{\chi_2} \cong \rho_1, \quad \text{CH}(k)e_{\chi_1} \cong \rho_2, \quad \text{CH}(k) \cong \rho_1 \oplus \rho_2.$$

Proof. The $\mathcal{G}_{\mathbb{Q}_p}$ -cosocle of $\rho_1 \oplus \rho_2$ is $\chi_2 \oplus \chi_1$. Since these characters are distinct, $\rho_1 \oplus \rho_2$ is a cyclic $k[[\mathcal{G}_{\mathbb{Q}_p}]]$ -module. Moreover, elements $g^2 - (\chi_1(g) + \chi_2(g))g + \chi_1 \chi_2(g)$ kill $\rho_1 \oplus \rho_2$, and hence we obtain a surjection of $\mathcal{G}_{\mathbb{Q}_p}$ -representations $\text{CH}(k) \twoheadrightarrow \rho_1 \oplus \rho_2$.

Since the order of H is prime to p , $\text{CH}(k)$ is semi-simple as an H -representation. If H acts on $v \in \text{CH}(k)$ by a character ψ , then for all $h \in H$, we have

$$0 = (h - \chi_1(h))(h - \chi_2(h))v = (\psi(h) - \chi_1(h))(\psi(h) - \chi_2(h))v.$$

Let $H_1 := \{h \in H : \psi(h) = \chi_1(h)\}$, $H_2 = \{h \in H : \psi(h) = \chi_2(h)\}$. Then H_1 and H_2 are subgroups of H , such that $H_1 \cup H_2 = H$. This implies, for example by calculating $|H|$ as $|H_1| + |H_2| - |H_1 \cap H_2|$ and $|H_1||H_2|/|H_1 \cap H_2|$, that either $H_1 = H$ or $H_2 = H$. Thus either $\psi = \chi_1$ or $\psi = \chi_2$.

Let I_P be the augmentation ideal in $k[[P]]$. Since P is normal, $\text{CH}(k)/I_P\text{CH}(k) \cong k[H]/\overline{J}$, where \overline{J} is the two-sided ideal generated by all the elements of the form $h^2 - (\chi_1(h) + \chi_2(h))h + \chi_1\chi_2(h)$, for all $h \in H$. We know that $k[H]/\overline{J}$ admits $(\rho_1 \oplus \rho_2)/I_P(\rho_1 \oplus \rho_2) \cong \chi_2 \oplus \chi_1$ as a quotient. Since H is abelian, χ_1 and χ_2 occur in $k[H]$ with multiplicity one. Moreover, the argument above shows that no other character can occur in $k[H]/\overline{J}$. Hence, we have an isomorphism of G -representations $\text{CH}(k)/I_P\text{CH}(k) \cong \chi_2 \oplus \chi_1$.

Topological Nakayama's lemma implies that $\text{CH}(k)$ is generated as $k[[P]]$ -module by the two elements e_{χ_1} and e_{χ_2} defined in the previous section. Let M_1 be the $k[[P]]$ -submodule of $\text{CH}(k)$ generated by e_{χ_2} , and let M_2 be the $k[[P]]$ -submodule of $\text{CH}(k)$ generated by e_{χ_1} , so that $M_1 = \text{CH}(k)e_{\chi_2}$ and $M_2 = \text{CH}(k)e_{\chi_1}$. We claim that $M_1 \cong \rho_1$ and $M_2 \cong \rho_2$ as G -representations. The claim implies that the surjection $\text{CH}(k) \twoheadrightarrow \rho_1 \oplus \rho_2$ is an isomorphism.

We will show the claim for M_1 , the proof for M_2 is the same. We know that $I_P M_1 / I_P^2 M_1$ as an H -representation is a direct sum of copies of χ_1 and χ_2 . Since $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_1)$ is one dimensional, χ_1 appears with multiplicity 1. If χ_2 appears in $I_P M_1 / I_P^2 M_1$, then M_1 would admit a quotient N , which is a non-split extension of χ_2 by itself as a G -representation. If $p \in P$ is such that p does not act trivially on N , and $h \in H$ is such that $\chi_1(h) \neq \chi_2(h)$ then the minimal polynomial of $g := hp$ acting on N is $(x - \chi_2(g))^2$. Since $(g - \chi_1(g))(g - \chi_2(g))$ kills $\text{CH}(k)$, it will also kill N . Since $\chi_1(g) \neq \chi_2(g)$, we get a contradiction. Hence, $M_1 / I_P^2 M_1 \cong \rho_1$. Since $I_P M_1 / I_P^2 M_1$ is a one dimensional k -vector space on which H acts by χ_1 , and $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2)$ is one dimensional, the same argument shows that if $I_P^2 M_1 \neq 0$, then $I_P^2 M_1 / I_P^3 M_1$ is one dimensional, and H acts on it by χ_2 . Hence, it is enough to show that $e_{\chi_2} I_P M_1 = 0$, since then Nakayama's lemma would imply that $I_P^2 M_1 = 0$. From $(gh)^2 - (\chi_1(h) + \chi_2(h))gh + \chi_1\chi_2(h) = 0$, we get that the following holds in $\text{CH}(k)$:

$$gh = (\chi_1(h) + \chi_2(h)) - \chi_1\chi_2(h)h^{-1}g^{-1}, \quad \forall h \in H, \quad \forall g \in P.$$

Then

$$\begin{aligned} e_{\chi_2} g e_{\chi_2} &= \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} \chi_2(h_1^{-1}) \chi_2(h_2^{-1}) h_1 g h_2 \\ &= \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} \chi_2(h_1^{-1}) \chi_2(h_2^{-1}) h_1 (\chi_1(h_2) + \chi_2(h_2)) \\ &\quad - \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} \chi_2(h_1^{-1}) \chi_1(h_2) h_1 h_2^{-1} g^{-1} \\ &= e_{\chi_2} - \frac{1}{|H|^2} \sum_{h_1, k \in H} \chi_2(h_1^{-1}) \chi_1(k^{-1} h_1) k g = e_{\chi_2}, \end{aligned} \tag{2}$$

where we have used the orthogonality of characters. Hence, $e_{\chi_2}(g-1)e_{\chi_2} = 0$ in $\text{CH}(k)$ for all $g \in P$, and so $e_{\chi_2}I_P M_1 = 0$. \square

Lemma 3.2. *There is an isomorphism of k -algebras:*

$$\text{CH}(k)^{\text{op}} \cong \text{End}_{\mathcal{G}_{\mathbb{Q}_p}}(\rho_1 \oplus \rho_2) \cong \begin{pmatrix} ke_{\chi_1} & k\Phi_{12} \\ k\Phi_{21} & ke_{\chi_2} \end{pmatrix},$$

where Φ_{12}, Φ_{21} are elements of $\text{CH}(k)$ such that $\{e_{\chi_1}, \Phi_{21}, e_{\chi_2}, \Phi_{12}\}$ is a basis of $\text{CH}(k)$ as a k -vector space and the following relations hold:

$$\Phi_{12} = e_{\chi_1}\Phi_{12} = \Phi_{12}e_{\chi_2}, \quad \Phi_{21} = e_{\chi_2}\Phi_{21} = \Phi_{21}e_{\chi_1}, \quad \Phi_{12}\Phi_{21} = \Phi_{21}\Phi_{12} = 0.$$

Let $c_{12}, c_{21} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow k$ be the functions, such that an element $g \in \mathcal{G}_{\mathbb{Q}_p}$ is mapped to $\chi_1(g)e_{\chi_1} + c_{12}(g)\Phi_{12} + c_{21}(g)\Phi_{21} + \chi_2(g)e_{\chi_2}$ under the natural map $k[[\mathcal{G}_{\mathbb{Q}_p}]] \rightarrow \text{CH}(k)$. Then c_{12}, c_{21} are 1-cocycles, such that the image of c_{12} in $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_1)$, and the image of c_{21} in $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2)$, span the respective vector space.

Proof. The multiplication on the right by elements of $\text{CH}(k)$ induces an injection of k -algebras $\text{CH}(k)^{\text{op}} \hookrightarrow \text{End}_{\mathcal{G}_{\mathbb{Q}_p}}(\text{CH}(k))$. Since $\text{CH}(k) \cong \rho_1 \oplus \rho_2$ by Lemma 3.1, we deduce that both $\text{CH}(k)$ and $\text{End}_{\mathcal{G}_{\mathbb{Q}_p}}(\text{CH}(k))$ are 4-dimensional k -vector spaces. Hence the injection is an isomorphism. Since $\text{CH}(k)e_{\chi_1} \cong \rho_2$ and $\text{CH}(k)e_{\chi_2} \cong \rho_1$ by Lemma 3.1, and the restrictions of ρ_1 and ρ_2 to H are isomorphic to $\chi_1 \oplus \chi_2$, we deduce that

$$e_{\chi_1}\text{CH}(k)e_{\chi_1}, \quad e_{\chi_2}\text{CH}(k)e_{\chi_1}, \quad e_{\chi_2}\text{CH}(k)e_{\chi_2}, \quad e_{\chi_1}\text{CH}(k)e_{\chi_2}$$

are all 1-dimensional k -vector spaces. The idempotents e_{χ_1}, e_{χ_2} are basis vectors of $e_{\chi_1}\text{CH}(k)e_{\chi_1}$ and $e_{\chi_2}\text{CH}(k)e_{\chi_2}$, respectively. We choose a basis element Φ_{21} of $e_{\chi_2}\text{CH}(k)e_{\chi_1}$ and a basis element Φ_{12} of $e_{\chi_1}\text{CH}(k)e_{\chi_2}$. It is immediate that the claimed relations are satisfied.

Let $\bar{g} := g + J$ be the image of $g \in \mathcal{G}_{\mathbb{Q}_p}$ in $\text{CH}(k)$. Since $\{e_{\chi_1}, \Phi_{21}, e_{\chi_2}, \Phi_{12}\}$ is a basis of $\text{CH}(k)$ as a k -vector space, we may write

$$\bar{g} = c_{11}(g)e_{\chi_1} + c_{12}(g)\Phi_{12}(g) + c_{21}(g)\Phi_{21}(g) + c_{22}(g)e_{\chi_2} \text{ with } c_{ij}(g) \in k.$$

The left action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\text{CH}(k)e_{\chi_1}$ factors through the action of $\text{CH}(k)$ and hence g acts as \bar{g} . The multiplication relations imply that $\bar{g}e_{\chi_1} = c_{11}(g)e_{\chi_1} + c_{21}(g)\Phi_{21}$ and $\bar{g}\Phi_{21} = c_{22}(g)\Phi_{21}$. Thus the left action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\text{CH}(k)e_{\chi_1}$ with respect to the basis $\{e_{\chi_1}, \Phi_{21}\}$ is given by $g \mapsto \begin{pmatrix} c_{11}(g) & 0 \\ c_{21}(g) & c_{22}(g) \end{pmatrix}$. Since Lemma 3.1 tells us that this representation is isomorphic to ρ_2 , which is a non-split extension of distinct characters, we deduce that $c_{11}(g) = \chi_1(g)$, $c_{22}(g) = \chi_2(g)$ and c_{21} is a 1-cocycle, whose image in $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2)$ corresponds to ρ_2 . Since ρ_2 is non-split, the image of c_{21} is non-zero. Since by assumption $\chi_1\chi_2^{-1} \neq \mathbf{1}, \omega^{\pm 1}$, $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2)$ is a 1-dimensional k -vector space and hence the image of c_{21} is a basis vector. The same argument with $\text{CH}(k)e_{\chi_2}$ instead of $\text{CH}(k)e_{\chi_1}$ proves the analogous assertion about c_{12} . \square

Lemma 3.3. *Let $(t, d) \in D^{\text{ps}}(A)$ and let $\text{CH}(A)$ be the corresponding Cayley–Hamilton algebra. Then $e_{\chi_1}\text{CH}(A)e_{\chi_1}$ and $e_{\chi_2}\text{CH}(A)e_{\chi_2}$ are free A -modules of rank 1 with generators e_{χ_1}, e_{χ_2} respectively.*

Proof. We will show the statement for χ_1 , the proof for χ_2 is the same. We have

$$e_{\chi_1} \text{CH}(A) e_{\chi_1} / \mathfrak{m}_A e_{\chi_1} \text{CH}(A) e_{\chi_1} = e_{\chi_1} \text{CH}(k) e_{\chi_1},$$

which is a 1-dimensional vector space spanned by e_{χ_1} by Lemma 3.2. Nakayama's lemma implies that $e_{\chi_1} \text{CH}(A) e_{\chi_1}$ is a cyclic A -module with generator e_{χ_1} . It is enough to construct a surjection of A -modules onto A .

Since t is continuous, we may extend it to a map of A -modules, $t : A[[\mathcal{G}_{\mathbb{Q}_p}]] \rightarrow A$. If $h, g \in \mathcal{G}_{\mathbb{Q}_p}$ then $t(h(g^2 - t(g)g + d(g))) = t(hg^2) - t(g)t(gh) + d(g)t(h) = 0$, using the property (iii) above. Hence, the map factors through $t : \text{CH}(A) \rightarrow A$. Since $t(e_{\chi_1}) \pmod{\mathfrak{m}_A} = \text{tr } \rho_2(e_{\chi_1}) = 1$, $t(e_{\chi_1})$ is a unit in A and the map is surjective. Hence, t induces a surjection of A -modules $e_{\chi_1} \text{CH}(A) e_{\chi_1}$ onto A . \square

Proposition 3.4. *Let $k[\varepsilon]$ be the dual numbers over k . There is an exact sequence of k -vector spaces:*

$$(3) \quad 0 \rightarrow \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_1) \oplus \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_2) \rightarrow D^{\text{ps}}(k[\varepsilon]) \rightarrow \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_2) \otimes \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_1).$$

Proof. The proof is essentially the same as [1, Thm. 2]. We supply the details, since in [1] the reference [2] is used, where some assumptions on p are made.

We will define the first non-trivial arrow in (3). Let D_{χ_1} and D_{χ_2} be the deformation functors of χ_1 and χ_2 , respectively. Sending $\{\tilde{\chi}_1, \tilde{\chi}_2\} \mapsto (\tilde{\chi}_1 + \tilde{\chi}_2, \tilde{\chi}_1 \tilde{\chi}_2)$ induces a map $D_{\chi_1}(k[\varepsilon]) \times D_{\chi_2}(k[\varepsilon]) \rightarrow D^{\text{ps}}(k[\varepsilon])$. We may recover $\tilde{\chi}_1$ and $\tilde{\chi}_2$ from t , as

$$\tilde{\chi}_1(g) = t(e_{\chi_1}g), \quad \tilde{\chi}_2(g) = t(e_{\chi_2}g).$$

Hence, the map is injective. The first arrow is obtained by identifying $D_{\chi_1}(k[\varepsilon])$ and $D_{\chi_2}(k[\varepsilon])$ with $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_1, \chi_1)$ and $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_2)$ respectively.

We will define the last arrow in (3). Let $(t, d) \in D^{\text{ps}}(k[\varepsilon])$ and let $\text{CH}(k[\varepsilon])$ be the corresponding Cayley–Hamilton algebra. Reducing modulo ε induces an isomorphism $\text{CH}(k[\varepsilon])/\varepsilon \text{CH}(k[\varepsilon]) \cong \text{CH}(k)$. Let $\tilde{\Phi}_{12}, \tilde{\Phi}_{21} \in \text{CH}(k[\varepsilon])$ be lifts of Φ_{12} and Φ_{21} such that $\tilde{\Phi}_{12} = e_{\chi_1} \tilde{\Phi}_{12} = \tilde{\Phi}_{12} e_{\chi_2}$, $\tilde{\Phi}_{21} = e_{\chi_2} \tilde{\Phi}_{21} = \tilde{\Phi}_{21} e_{\chi_1}$. Then $\tilde{\Phi}_{12}$ is a generator for $e_{\chi_1} \text{CH}(k[\varepsilon]) e_{\chi_2}$ and $\tilde{\Phi}_{21}$ is a generator of $e_{\chi_2} \text{CH}(k[\varepsilon]) e_{\chi_1}$ as an A -module. Since $e_{\chi_2} \tilde{\Phi}_{21} \tilde{\Phi}_{12} e_{\chi_2} = \tilde{\Phi}_{21} \tilde{\Phi}_{12}$, and $\tilde{\Phi}_{12} \tilde{\Phi}_{21} = 0$, there is a unique $\lambda_{t,d} \in k$, such that $\tilde{\Phi}_{21} \tilde{\Phi}_{12} = \varepsilon \lambda_{t,d} e_{\chi_2}$. Since $(1 + \varepsilon \mu) \tilde{\Phi}_{21} \tilde{\Phi}_{12} = (1 + \varepsilon \mu) \varepsilon \lambda_{t,d} e_{\chi_2} = \varepsilon \lambda_{t,d} e_{\chi_2}$, $\lambda_{t,d}$ does not depend on the choice of the lift. For all $g, h \in \mathcal{G}_{\mathbb{Q}_p}$ we have

$$e_{\chi_2} g e_{\chi_1} h e_{\chi_2} = \tilde{c}_{21}(g) \tilde{\Phi}_{21} \tilde{c}_{12}(h) \tilde{\Phi}_{21} = \varepsilon \lambda_{t,d} c_{21}(g) c_{12}(h) e_{\chi_2}$$

and hence

$$\varepsilon \lambda_{t,d} c_{21}(g) c_{12}(h) e_{\chi_2} = t(e_{\chi_2} g e_{\chi_1} h e_{\chi_2}).$$

The map $(t, d) \mapsto \lambda_{t,d} c_{21} c_{12}$ defines the last arrow.

Given a function $f : \mathcal{G}_{\mathbb{Q}_p} \rightarrow k[\varepsilon]$, we define $f_0, f_1 : G \rightarrow k$ by $f(g) = f_0(g) + \varepsilon f_1(g)$, so that $t_0 = \chi_1 + \chi_2$ and $d_0 = \chi_1 \chi_2$. The k -vector space structure in $D^{\text{ps}}(k[\varepsilon])$ is given by $\lambda(t, d) + \mu(t', d') = (t_0 + \varepsilon(\lambda t_1 + \mu t'_1), d_0 + \varepsilon(\lambda d_1 + \mu d'_1))$. Since $t(e_{\chi_2} g e_{\chi_1} h e_{\chi_2}) = \varepsilon t_1(e_{\chi_2} g e_{\chi_1} h e_{\chi_2})$, the last arrow in (3) is k -linear.

If $t = \tilde{\chi}_1 + \tilde{\chi}_2$ then using orthogonality of characters we get that $t(e_{\chi_2} g e_{\chi_1} h e_{\chi_2}) = 0$, for all $g, h \in \mathcal{G}_{\mathbb{Q}_p}$. Hence (3) is a complex.

If (t, d) is mapped to zero, then $\tilde{\Phi}_{21} \tilde{\Phi}_{12} = 0$. Hence, the $k[\varepsilon]$ -module generated by $\tilde{\Phi}_{12}$ is stable under the action of $\text{CH}(k[\varepsilon])$. It follows from Lemma 3.3 that the quotient $\text{CH}(k[\varepsilon]) e_{\chi_2} / k[\varepsilon] \tilde{\Phi}_{12}$ is a free $k[\varepsilon]$ -module of rank 1 generated by e_{χ_2} .

The group $\mathcal{G}_{\mathbb{Q}_p}$ acts on the module by the character $\tilde{\chi}_2 : \mathcal{G}_{\mathbb{Q}_p} \rightarrow k[\varepsilon]^\times$, which is a deformation of χ_2 . Since $g^2 - t(g)g + d(g)$ will kill the module, we obtain that $\tilde{\chi}_2(g)^2 - t(g)\tilde{\chi}_2(g) + d(g) = 0$ in $k[\varepsilon]$. Hence, $t(g) = \tilde{\chi}_2(g) + d(g)\tilde{\chi}_2(g)^{-1}$. Since $d\tilde{\chi}_2^{-1}$ is a deformation of χ_1 , we deduce that (3) is exact. \square

Lemma 3.5. *If $p = 2$ then the last arrow in (3) is zero. Hence, $\dim_k D^{\text{ps}}(k[\varepsilon]) = 6$.*

Proof. The cup product induces a non-degenerate H -equivariant alternating pairing

$$(4) \quad H^1(P, k) \times H^1(P, k) \xrightarrow{\cup} H^2(P, k).$$

Since $p = 2$, the cyclotomic character modulo p is trivial, and hence H acts trivially on $H^2(P, k)$. Since the order of H is prime to p , for any character ψ of H , $H^1(G, \psi)$ is the ψ -isotypic subspace of $H^1(P, k)$. Since $\chi_1\chi_2^{-1} \neq \mathbf{1}$ both $H^1(G, \chi_1\chi_2^{-1})$ and $H^1(G, \chi_1^{-1}\chi_2)$ are one dimensional, and the pairing is non-degenerate and H -equivariant, (4) induces an isomorphism:

$$(5) \quad H^1(G, \chi_1\chi_2^{-1}) \otimes H^1(G, \chi_1^{-1}\chi_2) \xrightarrow{\cong} H^2(G, k).$$

By interpreting the cup product as Yoneda pairing, we deduce from (5) that there does not exist a representation τ of G , such that the socle and the cosocle of τ is isomorphic to χ_2 and the semi-simplification is isomorphic to $\chi_2 \oplus \chi_1 \oplus \chi_2$.

If the last arrow in (3) was non-zero, then we could construct such τ as follows: let $\text{CH}(k[\varepsilon])$ be the Cayley-Hamilton algebra, which corresponds to a pair $(t, d) \in D^{\text{ps}}(k[\varepsilon])$, which does not map to zero under the last arrow in (3). Then $\text{CH}(k[\varepsilon])e_{\chi_2}/\varepsilon\Phi_{12}$ would be such representation. \square

Sending a representation ρ to the pair $(\text{tr } \rho, \det \rho)$ induces a natural transformations $D_1 \rightarrow D^{\text{ps}}$, $D_2 \rightarrow D^{\text{ps}}$, and hence homomorphisms of local \mathcal{O} -algebras $t_1 : R^{\text{ps}} \rightarrow R_1$, $t_2 : R^{\text{ps}} \rightarrow R_2$.

Proposition 3.6. *Sending a representation ρ to the pair $(\text{tr } \rho, \det \rho)$ induces isomorphisms between the local \mathcal{O} -algebras $t_1 : R^{\text{ps}} \xrightarrow{\cong} R_1$, $t_2 : R^{\text{ps}} \xrightarrow{\cong} R_2$.*

Proof. If $p > 2$ this is shown in [14, Prop.B.17], using [1, Thm. 2] and the fact that R_1 and R_2 are formally smooth as an input. We assume that $p = 2$. Chenevier has shown in [7, Cor.4.4] that the maps are surjective and become isomorphisms after inverting 2. The result follows by combining Chenevier's argument with Lemma 3.5 as we now explain. The group homomorphisms $\det \rho_1^{\text{univ}} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow R_1^\times$ factors through the maximal abelian quotient of $\mathcal{G}_{\mathbb{Q}_p}$. Composing it with the Artin map of local class field theory, we obtain a continuous group homomorphism $\mathbb{Q}_2^\times \rightarrow R_1^\times$. The restriction of this map to the subgroup $\mu := \{\pm 1\} \subset \mathbb{Q}_2^\times$ induces a homomorphism of \mathcal{O} -algebras $\Lambda \rightarrow R_1$, where $\Lambda := \mathcal{O}[\mu]$ is the group algebra of μ over \mathcal{O} . The same argument with d^{univ} instead of $\det \rho_1^{\text{univ}}$, also makes R^{ps} into a Λ -algebra. Moreover, it is immediate that t_1 is a homomorphism of Λ -algebras. Chenevier shows that R_1 is formally smooth over Λ of dimension 5, and that t_1 is surjective. He proves this last assertion by checking that the map $D_1(k[\varepsilon]) \rightarrow D^{\text{ps}}(k[\varepsilon])$ is injective. It follows from Lemma 3.5, that both have the same dimension as k -vector space, thus the map is bijective. Since t_1 is a map of Λ -algebras and R_1 is formally smooth over Λ , we conclude that $R^{\text{ps}} \cong R_1$. In particular, $R^{\text{ps}} \cong \Lambda[[x_1, \dots, x_5]]$, where $\Lambda \cong \mathcal{O}[[y]]/((1+y)^2 - 1)$. \square

Lemma 3.7. *The restriction of $(t^{\text{univ}}, d^{\text{univ}})$ to H is equal to $([\chi_1] + [\chi_2], [\chi_1\chi_2])$, where square brackets denote the Teichmüller lift to \mathcal{O} .*

Proof. Let D_H^{ps} be the deformation functor parameterizing 2-dimensional determinants $(d, t) : H \rightarrow A$ lifting $(\chi_1 + \chi_2, \chi_1\chi_2)$. We claim that the corresponding deformation ring R_H^{ps} is equal to \mathcal{O} , and $([\chi_1] + [\chi_2], [\chi_1\chi_2])$ is the universal 2-dimensional determinant. The claim implies the assertion of the lemma, as it follows from the proof of Proposition 3.6 that R^{ps} is a flat \mathcal{O} -algebra. To prove the claim it is enough to verify that $D_H^{\text{ps}}(k[\varepsilon]) = D_H^{\text{ps}}(k)$, since $([\chi_1] + [\chi_2], [\chi_1\chi_2])$ gives a characteristic zero point. If $(t, d) \in D_H^{\text{ps}}(k[\varepsilon])$ let $\text{CH}(k[\varepsilon])$ be the corresponding Cayley–Hamilton algebra. We have already shown in the first part of the proof of Lemma 3.1 that $\text{CH}(k[\varepsilon])/\varepsilon\text{CH}(k[\varepsilon])$ is a 2-dimensional k -vector space with basis $\{e_{\chi_1}, e_{\chi_2}\}$. Nakayama’s lemma and Lemma 3.3 implies that $\text{CH}(k[\varepsilon])$ is a free $k[\varepsilon]$ -module with basis $\{e_{\chi_1}, e_{\chi_2}\}$. Since H acts on e_{χ_1} by the character χ_1 and on e_{χ_2} by the character χ_2 we deduce that $\text{CH}(k[\varepsilon]) \cong \text{CH}(k) \otimes_k k[\varepsilon]$ as $k[\varepsilon][H]$ -modules. This implies that $(t, d) = (\chi_1 + \chi_2, \chi_1\chi_2)$. \square

Proposition 3.8. *There is an ideal \mathfrak{r} of R^{ps} uniquely determined by the following universal property: an ideal J of R^{ps} contains \mathfrak{r} if and only if $t^{\text{univ}} \pmod{J} = \psi_1 + \psi_2$, $d^{\text{univ}} \pmod{J} = \psi_1\psi_2$, where $\psi_1, \psi_2 : \mathcal{G}_{\mathbb{Q}_p} \rightarrow R^{\text{ps}}/J$, are deformations of χ_1 and χ_2 , respectively, to R^{ps}/J . The ring $R^{\text{ps}}/\mathfrak{r}$ is reduced and \mathcal{O} -torsion free. The ideal \mathfrak{r} is principal, generated by a regular element.*

Proof. If $p > 2$ this is proved in [14, §B.2]. The proof is essentially the same for $p = 2$. If $\chi : \mathcal{G}_{\mathbb{Q}_p} \rightarrow k^\times$ is a continuous character then its deformation problem D_χ is pro-represented by R_χ , which is isomorphic to the completed group algebra over \mathcal{O} of the pro-2 completion of \mathbb{Q}_2^\times , so that $R_\chi \cong \mathcal{O}[[x, y, z]]/((1+y)^2 - 1)$. Since $\chi_1 \neq \chi_2$, using orthogonality of characters, one shows that for each $A \in \mathfrak{A}$ the map $D_{\chi_1\chi_2}(A) \times D_\chi(A) \rightarrow D^{\text{ps}}(A)$, $(d, \psi_1) \mapsto (d\psi_1^{-1} + \psi_1, d)$ is injective. We thus obtain a surjection of Λ -algebras $R^{\text{ps}} \twoheadrightarrow R_{\chi_1\chi_2} \widehat{\otimes}_{\mathcal{O}} R_{\chi_1}$. The ideal \mathfrak{r} is precisely the kernel of this map. Since the rings are isomorphic to $\Lambda[[x_1, \dots, x_5]]$ and $\Lambda[[x_1, z_1, x_2, y_2, z_2]]/((1+y_2)^2 - 1)$, respectively, this allows us to conclude. \square

We will refer to \mathfrak{r} as the reducibility ideal, and to $V(\mathfrak{r})$ as the reducibility locus.

Let $(t^{\text{univ}}, d^{\text{univ}}) \in D^{\text{ps}}(R^{\text{ps}})$ be the universal object. Let J be the closed two-sided ideal in $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ generated by all the elements of the form $g^2 - t^{\text{univ}}(g)g + d^{\text{univ}}(g)$, for all $g \in \mathcal{G}_{\mathbb{Q}_p}$, and let

$$\text{CH}(R^{\text{ps}}) := R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]/J.$$

Proposition 3.9. *The isomorphisms $t_1 : R^{\text{ps}} \cong R_1$, $t_2 : R^{\text{ps}} \cong R_2$ induce isomorphisms of left $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ -modules:*

$$\text{CH}(R^{\text{ps}})e_{\chi_2} \cong \rho_1^{\text{univ}}, \quad \text{CH}(R^{\text{ps}})e_{\chi_1} \cong \rho_2^{\text{univ}}, \quad \text{CH}(R^{\text{ps}}) \cong \rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}.$$

Proof. Since $\rho_1 \oplus \rho_2$ is a cyclic $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ -module, there is a map of $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ -modules $\phi : R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]] \rightarrow \rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}$, such that after composing with the reduction modulo the maximal ideal of R^{ps} , we obtain a surjection $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]] \twoheadrightarrow \rho_1 \oplus \rho_2$. Let C and K be the cokernel and the kernel of ϕ , respectively. Lemma 3.1 implies that $k \widehat{\otimes} \phi$ is an isomorphism between $k \widehat{\otimes}_{R^{\text{ps}}} \text{CH}(R^{\text{ps}})$ and $k \widehat{\otimes}_{R^{\text{ps}}} (\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$. This implies that $k \widehat{\otimes}_{R^{\text{ps}}} C = 0$ and Nakayama’s lemma for pseudo-compact R^{ps} -modules implies that $C = 0$. Thus ϕ is surjective. Since $\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}$ is a free R^{ps} -module of

rank 4, we deduce that $k \widehat{\otimes}_{R^{\text{ps}}} K = 0$, and so $K = 0$. Hence, ϕ is an isomorphism. The same argument proves the other assertions. \square

Corollary 3.10. *There is a natural isomorphism:*

$$\text{End}_{R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}) \cong \text{CH}(R^{\text{ps}})^{\text{op}}.$$

Proof. If J is a two-sided ideal in a ring A then multiplication on the right induces an isomorphism between $\text{End}_A(A/J)$ and the algebra opposite to A/J . The assertion follows from the last isomorphism in Proposition 3.9. \square

Remark 3.11. One may check that $g \mapsto d^{\text{univ}}(g)g^{-1}$ induces an involution on $\text{CH}(R^{\text{ps}})$ and hence an isomorphism of R^{ps} -algebras between $\text{CH}(R^{\text{ps}})$ and $\text{CH}(R^{\text{ps}})^{\text{op}}$.

Proposition 3.12. *The algebra $\text{CH}(R^{\text{ps}})$ is a free R^{ps} -module of rank 4:*

$$\text{CH}(R^{\text{ps}}) \cong \begin{pmatrix} R^{\text{ps}}e_{\chi_1} & R^{\text{ps}}\tilde{\Phi}_{12} \\ R^{\text{ps}}\tilde{\Phi}_{21} & R^{\text{ps}}e_{\chi_2} \end{pmatrix}.$$

The generators satisfy the following relations

$$(6) \quad e_{\chi_1}^2 = e_{\chi_1}, \quad e_{\chi_2}^2 = e_{\chi_2}, \quad e_{\chi_1}e_{\chi_2} = e_{\chi_2}e_{\chi_1} = 0,$$

$$(7) \quad e_{\chi_1}\tilde{\Phi}_{12} = \tilde{\Phi}_{12}e_{\chi_2} = \tilde{\Phi}_{12}, \quad e_{\chi_2}\tilde{\Phi}_{21} = \tilde{\Phi}_{21}e_{\chi_1} = \tilde{\Phi}_{21},$$

$$(8) \quad e_{\chi_2}\tilde{\Phi}_{12} = \tilde{\Phi}_{12}e_{\chi_1} = e_{\chi_1}\tilde{\Phi}_{21} = \tilde{\Phi}_{21}e_{\chi_2} = \tilde{\Phi}_{12}^2 = \tilde{\Phi}_{21}^2 = 0,$$

$$(9) \quad \tilde{\Phi}_{12}\tilde{\Phi}_{21} = ce_{\chi_1}, \quad \tilde{\Phi}_{21}\tilde{\Phi}_{12} = ce_{\chi_2}.$$

The element c generates the reducibility ideal in R^{ps} . In particular, c is R^{ps} -regular.

Proof. The last isomorphism in Proposition 3.9 and Proposition 3.6 imply that $\text{CH}(R^{\text{ps}})$ is a free R^{ps} -module of rank 4. Nakayama's lemma implies that any four elements of $\text{CH}(R^{\text{ps}})$, which map to a k -basis of $k \otimes_{R^{\text{ps}}} \text{CH}(R^{\text{ps}}) \cong \text{CH}(k)$ is an R^{ps} -basis of $\text{CH}(R^{\text{ps}})$. We lift the k -basis of $\text{CH}(k)$, described in Lemma 3.2 as follows. Let $\Psi_1, \Psi_2 \in \text{CH}(R^{\text{ps}})$ be any lifts of Φ_{12} and Φ_{21} , respectively. Let $\tilde{\Phi}_{12} = e_{\chi_1}\Psi_1e_{\chi_2}$, $\tilde{\Phi}_{21} = e_{\chi_2}\Psi_2e_{\chi_1}$. It follows from relations in Lemma 3.2 that $\tilde{\Phi}_{12}$ maps to Φ_{12} , $\tilde{\Phi}_{21}$ maps to Φ_{21} , hence $\{e_{\chi_1}, \tilde{\Phi}_{12}, \tilde{\Phi}_{21}, e_{\chi_2}\}$ is an R^{ps} -basis of $\text{CH}(R^{\text{ps}})$. The relations in (6), (7), (8) follow from the fact that e_{χ_1} and e_{χ_2} are orthogonal idempotents. Since $e_{\chi_1}\tilde{\Phi}_{12}\tilde{\Phi}_{21}e_{\chi_1} = \tilde{\Phi}_{12}\tilde{\Phi}_{21}$, and $e_{\chi_2}\tilde{\Phi}_{21}\tilde{\Phi}_{12}e_{\chi_2} = \tilde{\Phi}_{21}\tilde{\Phi}_{12}$, we deduce that there are $c_1, c_2 \in R^{\text{ps}}$, such that

$$(10) \quad \tilde{\Phi}_{12}\tilde{\Phi}_{21} = c_1e_{\chi_1}, \quad \tilde{\Phi}_{21}\tilde{\Phi}_{12} = c_2e_{\chi_2}.$$

Now $t^{\text{univ}} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow R^{\text{ps}}$ extends to a map of R^{ps} -modules $t^{\text{univ}} : \text{CH}(R^{\text{ps}}) \rightarrow R^{\text{ps}}$. For each $a, b \in \text{CH}(R^{\text{ps}})$ we have $t^{\text{univ}}(ab) = t^{\text{univ}}(ba)$ since this identity holds for $a, b \in R^{\text{ps}}[\mathcal{G}_{\mathbb{Q}_p}]$, and the image of $R^{\text{ps}}[\mathcal{G}_{\mathbb{Q}_p}]$ in $\text{CH}(R^{\text{ps}})$ is dense, and t^{univ} is continuous. It follows from Lemma 3.7 that $t^{\text{univ}}(e_{\chi_1}) = t^{\text{univ}}(e_{\chi_2}) = 1$. This together with (10) implies that $c_1 = t^{\text{univ}}(\tilde{\Phi}_{12}\tilde{\Phi}_{21}) = t^{\text{univ}}(\tilde{\Phi}_{21}\tilde{\Phi}_{12}) = c_2 =: c$.

We will show that c generates the reducibility ideal in R^{ps} . It then will follow from the last part of Proposition 3.8 that c is regular. Since $R^{\text{ps}}/\mathfrak{r}$ is reduced and \mathcal{O} -torsion free by Proposition 3.8, if the image of c in $R^{\text{ps}}/\mathfrak{r}$ is non-zero then there is a maximal ideal \mathfrak{n} of $(R^{\text{ps}}/\mathfrak{r})[1/p]$, which does not contain c . Since $\mathcal{O}_{\kappa(\mathfrak{n})}[[\mathcal{G}_{\mathbb{Q}_p}]] \cong \mathcal{O}_{\kappa(\mathfrak{n})} \widehat{\otimes}_{R^{\text{ps}}} R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$, the images of $\mathcal{O}_{\kappa(\mathfrak{n})}[[\mathcal{G}_{\mathbb{Q}_p}]]$ and $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ in

$\text{End}_{\mathcal{O}_{\kappa(\mathfrak{n})}}(\text{CH}(R^{\text{ps}})e_{\chi_1} \otimes_{R^{\text{ps}}} \mathcal{O}_{\kappa(\mathfrak{n})})$ coincide. In particular it will contain the images of $\tilde{\Phi}_{12}, \tilde{\Phi}_{21}, e_{\chi_1}, e_{\chi_2}$. The action of these elements on $\text{CH}(R^{\text{ps}})e_{\chi_1}$ with respect to the R^{ps} -basis $e_{\chi_1}, \tilde{\Phi}_{21}$ is given by the matrices $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Since the image of c in $\kappa(\mathfrak{n})$ is non-zero this implies that the map $\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]] \otimes_{\mathcal{O}} \kappa(\mathfrak{n}) \rightarrow \text{End}_{\kappa(\mathfrak{n})}(\text{CH}(R^{\text{ps}})e_{\chi_1} \otimes_{R^{\text{ps}}} \kappa(\mathfrak{n}))$ is surjective and hence $\rho_2^{\text{univ}} \otimes_{R^{\text{ps}}} \kappa(\mathfrak{n}) \cong \text{CH}(R^{\text{ps}})e_{\chi_1} \otimes_{R^{\text{ps}}} \kappa(\mathfrak{n})$ is an irreducible representation of $\mathcal{G}_{\mathbb{Q}_p}$. Moreover, its trace is equal to the specialization of t^{univ} at \mathfrak{n} . This leads to a contradiction as \mathfrak{n} contains the reducibility ideal. Hence, $c \in \mathfrak{r}$. For the other inclusion we observe that (7), (9) imply that the R^{ps}/c -submodule of $\rho_1^{\text{univ}}/(c)$ generated by $\tilde{\Phi}_{21}$ is stable under the action of $\text{CH}(R^{\text{ps}})$. Moreover, it is free over R^{ps}/c of rank 1. It follows from the definition of the reducible locus that the homomorphism $R^{\text{ps}} \twoheadrightarrow R^{\text{ps}}/c$ factors through $R^{\text{ps}}/\mathfrak{r}$. \square

4. THE CENTRE

In this section we compute the ring $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$ and show that its centre is naturally isomorphic to R^{ps} . This result is used in [16] to show that the centre of a certain category of $\text{GL}_2(\mathbb{Q}_2)$ representations is naturally isomorphic to a quotient of R^{ps} , corresponding to a fixed determinant, in the same way as [14, Prop. B.26] is used in [14, Cor.8.11].

Lemma 4.1. *Let F be a finite extension of \mathbb{Q}_p , let \mathcal{G}_F be its absolute Galois group, and let $\rho : \mathcal{G}_F \rightarrow \text{GL}_n(k)$ be a continuous representation, such that $\text{End}_{\mathcal{G}_F}(\rho) = k$. Let ρ^{univ} be the universal deformation of ρ and let R be the universal deformation ring. Then for pseudo-compact R -modules m_1, m_2 , the functor $m \mapsto m \hat{\otimes}_R \rho^{\text{univ}}$ induces an isomorphism*

$$\text{Hom}_R^{\text{cont}}(m_1, m_2) \xrightarrow{\cong} \text{Hom}_{\mathcal{O}[[\mathcal{G}_F]]}^{\text{cont}}(m_1 \hat{\otimes}_R \rho^{\text{univ}}, m_2 \hat{\otimes}_R \rho^{\text{univ}}).$$

In particular, $\text{End}_{\mathcal{O}[[\mathcal{G}_F]]}^{\text{cont}}(\rho^{\text{univ}}) \cong \text{End}_{R[[\mathcal{G}_F]]}(\rho^{\text{univ}}) \cong R$.

Proof. We argue as in [14, Lem.11.5, Cor.11.6] by induction on $\ell(m_1) + \ell(m_2)$ that for finite length modules m_1, m_2 of R the functor $m \mapsto m \otimes_R \rho^{\text{univ}}$ induces an isomorphism

$$\text{Hom}_R(m_1, m_2) \xrightarrow{\cong} \text{Hom}_{\mathcal{O}[[\mathcal{G}_F]]}(m_1 \otimes_R \rho^{\text{univ}}, m_2 \otimes_R \rho^{\text{univ}})$$

and an injection

$$\text{Ext}_R^1(m_1, m_2) \hookrightarrow \text{Ext}_{\mathcal{O}[[\mathcal{G}_F]]}^1(m_1 \otimes_R \rho^{\text{univ}}, m_2 \otimes_R \rho^{\text{univ}}).$$

This last map is well defined in terms of Yoneda extensions, because ρ^{univ} is flat over R . The induction step follows by looking at long exact sequences, as in the proof of [13, Lem. A.1].

To start the induction we need to check that the statement is true for $m_1 = m_2 = k$. The assertion about homomorphisms in this case, comes from the assumption that $\text{End}_{\mathcal{G}_F}(\rho) = k$. Consider an extension of R -modules, $0 \rightarrow k \rightarrow m \rightarrow k \rightarrow 0$. If the corresponding extension $0 \rightarrow \rho \rightarrow m \otimes_R \rho^{\text{univ}} \rightarrow \rho \rightarrow 0$ is split, then m is killed by ϖ . In which case, the map $m \twoheadrightarrow m/k \hookrightarrow m$ makes m into a free rank 1 module over the dual numbers $k[\varepsilon]$, and hence induces a homomorphism $\phi : R \rightarrow k[\varepsilon]$ in \mathfrak{A} . A standard argument in deformation theory shows that mapping ϕ to the equivalence class of $k[\varepsilon] \otimes_{R, \phi} \rho^{\text{univ}}$ induces a bijection between $\text{Hom}(R, k[\varepsilon])$ and

$\text{Ext}_{k[\mathcal{G}_F]}^1(\rho, \rho)$. Thus if $\mathfrak{m} \otimes_R \rho^{\text{univ}} \cong \rho \oplus \rho$ as an $\mathcal{O}[\mathcal{G}_F]$ -module then $\mathfrak{m} \cong k \oplus k$ and so the map $\text{Ext}_R^1(k, k) \rightarrow \text{Ext}_{\mathcal{O}[\mathcal{G}_F]}^1(\rho, \rho)$ is injective.

The general case follows by writing pseudo-compact modules as a projective limit of modules of finite length, see the proof of [14, Cor.11.6]. The last assertion of the proposition follows by taking $\mathfrak{m}_1 = \mathfrak{m}_2 = R$. \square

Remark 4.2. There is a gap in the proof of [14, Lem.B.21]. The issue is that the ring denoted by $\text{End}_G(\tilde{\rho}_{ij})$ there is equal to $\text{End}_{\mathcal{O}}^{\text{cont}}(R) \times \text{End}_{\mathcal{O}}^{\text{cont}}(R)$, which is much bigger than $R \times R$.

Proposition 4.3.

$$\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}) \cong \text{End}_{R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}}) \cong \text{CH}(R^{\text{ps}})^{\text{op}}.$$

Proof. The second isomorphism is given by Corollary 3.10. To establish the first isomorphism, it is enough to show that the injection

$$(11) \quad \text{Hom}_{R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]}(\rho_i^{\text{univ}}, \rho_j^{\text{univ}}) \hookrightarrow \text{Hom}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_i^{\text{univ}}, \rho_j^{\text{univ}}),$$

is an isomorphism for all $i, j \in \{1, 2\}$. If $i = j$ then the assertion follows from Lemma 4.1.

We will prove that (11) is an isomorphism if $i = 1, j = 2$. For a finitely generated $R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ -module M we let

$$A(M) := \text{Hom}_{R^{\text{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]}(\rho_1^{\text{univ}}, M), \quad B(M) := \text{Hom}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}}, M).$$

We have a natural inclusion $A(M) \hookrightarrow B(M)$, which is an isomorphism if $M = \rho_1^{\text{univ}}$ by Lemma 4.1. It follows from the description of ρ_1^{univ} and ρ_2^{univ} in Proposition 3.9 and Proposition 3.12 that multiplication by $\tilde{\Phi}_{21}$ on the right induces an injection $\rho_2^{\text{univ}} \hookrightarrow \rho_1^{\text{univ}}$. We denote the quotient by Q . We apply A and B to the exact sequence to get a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A(\rho_2^{\text{univ}}) & \longrightarrow & A(\rho_1^{\text{univ}}) & \longrightarrow & A(Q) \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & B(\rho_2^{\text{univ}}) & \longrightarrow & B(\rho_1^{\text{univ}}) & \longrightarrow & B(Q). \end{array}$$

The diagram implies that the first vertical arrow is an isomorphism. The same argument with ρ_1^{univ} instead of ρ_2^{univ} and $\tilde{\Phi}_{12}$ instead of $\tilde{\Phi}_{21}$ shows that (11) is an isomorphism for $i = 2, j = 1$. \square

Corollary 4.4. *The center of $\text{End}_{\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]]}^{\text{cont}}(\rho_1^{\text{univ}} \oplus \rho_2^{\text{univ}})$ is naturally isomorphic to R^{ps} .*

Proof. Proposition 4.3 implies that it is enough to compute the center of $\text{CH}(R^{\text{ps}})^{\text{op}}$. An element $\Upsilon \in \text{CH}(R^{\text{ps}})$ maybe expressed uniquely as $a_{11}e_{\chi_1} + a_{12}\tilde{\Phi}_{12} + a_{21}\tilde{\Phi}_{21} + a_{22}e_{\chi_2}$ with $a_{11}, a_{12}, a_{21}, a_{22} \in R^{\text{ps}}$. If Υ lies in the centre it must commute with e_{χ_1}, e_{χ_2} and $\tilde{\Phi}_{12}$. Using (7), (9) we deduce that $a_{12} = a_{21} = 0$ and $ca_{11} = ca_{22}$. It follows from 3.12 that c is a regular element, thus $a_{11} = a_{22}$, and since $e_{\chi_1} + e_{\chi_2}$ is the identity on $\text{CH}(R^{\text{ps}})$ we deduce that $\Upsilon \in R^{\text{ps}}$. On the other hand R^{ps} is contained in the center of $\text{CH}(R^{\text{ps}})$ by construction. \square

Remark 4.5. If the determinant is fixed throughout then one may show that the composition $\mathcal{O}[[\mathcal{G}_{\mathbb{Q}_p}]] \rightarrow R^{\text{ps}, \psi}[[\mathcal{G}_{\mathbb{Q}_p}]] \twoheadrightarrow \text{CH}(R^{\text{ps}, \psi})$ is surjective. This can be used

to give another proof of the results in this section, in the case when the determinant is fixed.

5. VERSAL DEFORMATION RING

In this section we compute the versal deformation ring of the representation $\rho = \chi_1 \oplus \chi_2$. Recall, [12], that a lift of ρ to $A \in \mathfrak{A}$ is a continuous representation $\mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(A)$ congruent to ρ modulo the maximal ideal of A . Two lifts are equivalent if they are conjugate by a matrix lying in the kernel of $\mathrm{GL}_2(A) \rightarrow \mathrm{GL}_2(k)$. Let $D^{\mathrm{ver}} : \mathfrak{A} \rightarrow \mathrm{Sets}$ be the functor which sends A to the set of equivalence classes of lifts of ρ to A . We define

$$(12) \quad R^{\mathrm{ver}} := R^{\mathrm{ps}}[[x, y]]/(xy - c),$$

where $c \in R^{\mathrm{ps}}$ is defined in (9). The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $\mathrm{End}_{R^{\mathrm{ver}}}(R^{\mathrm{ver}} \oplus R^{\mathrm{ver}})$ satisfy the same relations as e_{χ_1} , $\tilde{\Phi}_{12}$, $\tilde{\Phi}_{21}$, e_{χ_2} in $\mathrm{CH}(R^{\mathrm{ps}})$, see (6), (7), (8), (9). Thus mapping

$$e_{\chi_1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Phi}_{12} \mapsto \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \quad \tilde{\Phi}_{21} \mapsto \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}, \quad e_{\chi_2} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

induces a homomorphism of R^{ps} -algebras $\mathrm{CH}(R^{\mathrm{ps}}) \rightarrow \mathrm{End}_{R^{\mathrm{ver}}}(R^{\mathrm{ver}} \oplus R^{\mathrm{ver}})$. By composing it with a natural map $\mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathrm{CH}(R^{\mathrm{ps}})$ we obtain a representation

$$\rho^{\mathrm{ver}} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(R^{\mathrm{ver}}).$$

It is immediate that $\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}} k \cong \chi_1 \oplus \chi_2$. For $A \in \mathfrak{A}$ we let

$$h^{\mathrm{ver}}(A) := \mathrm{Hom}_{\mathfrak{A}}(R^{\mathrm{ver}}, A) := \varinjlim_n \mathrm{Hom}_{\mathfrak{A}}(R^{\mathrm{ver}}/\mathfrak{m}^n, A),$$

where \mathfrak{m} is the maximal ideal of R^{ver} . Mapping $\varphi \in h^{\mathrm{ver}}(A)$ to the equivalence class of $\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}, \varphi} A$ induces a natural transformation

$$(13) \quad \alpha : h^{\mathrm{ver}} \rightarrow D^{\mathrm{ver}}.$$

We define an equivalence relation on $h^{\mathrm{ver}}(A)$, by the rule $\varphi_1 \sim \varphi_2$ if φ_1 and φ_2 agree on R^{ps} and there is $\lambda \in 1 + \mathfrak{m}_A$, such that $\varphi_1(x) = \lambda\varphi_2(x)$, $\varphi_1(y) = \lambda^{-1}\varphi_2(y)$.

Lemma 5.1. *For all $A \in \mathfrak{A}$ the natural transformation α induces a bijection between $h^{\mathrm{ver}}(A)/\sim$ and $D^{\mathrm{ver}}(A)$.*

Proof. We first observe that if $\varphi_1 \sim \varphi_2$ then the representations $\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}, \varphi_1} A$, $\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}, \varphi_2} A$ are conjugate by a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, for some $\lambda \in 1 + \mathfrak{m}_A$. Hence, $\alpha(\varphi_1) = \alpha(\varphi_2)$, and the map is well defined.

If $\varphi_1, \varphi_2 \in h^{\mathrm{ver}}(A)$ are such that $\alpha(\varphi_1) = \alpha(\varphi_2)$, then there is a matrix $M \in \mathrm{GL}_2(A)$, congruent to the identity modulo \mathfrak{m}_A , such that

$$\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}, \varphi_1} A = M(\rho^{\mathrm{ver}} \otimes_{R^{\mathrm{ver}}, \varphi_2} A)M^{-1}.$$

Hence the representations have the same trace and determinant, which implies that φ_1 and φ_2 agree on R^{ps} . Moreover, since both representations map $h \in H$ to a matrix $\begin{pmatrix} [\chi_1](h) & 0 \\ 0 & [\chi_2](h) \end{pmatrix}$, M has to commute with the image of H . This implies that M is a diagonal matrix, and hence $\varphi_1 \sim \varphi_2$. Thus the map is injective.

Let $\rho_A : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(A)$ be a lift of ρ . Since $(\mathrm{tr} \rho_A, \det \rho_A) \in D^{\mathrm{ps}}(A)$, we obtain a map $\varphi : R^{\mathrm{ps}} \rightarrow A$. This allows us to view ρ_A as an $R^{\mathrm{ps}}[[\mathcal{G}_{\mathbb{Q}_p}]]$ -module, and by Cayley–Hamilton, as an $\mathrm{CH}(R^{\mathrm{ps}})$ -module. In other words we obtain a homomorphism of R^{ps} -algebras $\rho_A : \mathrm{CH}(R^{\mathrm{ps}}) \rightarrow \mathrm{End}_A(A \oplus A)$. We may conjugate ρ_A with $M \in \mathrm{GL}_2(A)$, which is congruent to 1 modulo \mathfrak{m}_A , such that every $h \in H$

is mapped to a matrix $\begin{pmatrix} [\chi_1](h) & 0 \\ 0 & [\chi_2](h) \end{pmatrix}$. Thus $\rho_A(e_{\chi_1}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_A(e_{\chi_2}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from (7) that there are $a_{12}, a_{21} \in \mathfrak{m}_A$, such that $\rho_A(\tilde{\Phi}_{12}) = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}$, $\rho_A(\tilde{\Phi}_{21}) = \begin{pmatrix} 0 & 0 \\ a_{21} & 0 \end{pmatrix}$. It follows from (9) that $a_{12}a_{21} = \varphi(c)$. Hence, we may extend $\varphi : R^{\text{ps}} \rightarrow A$ to R^{ver} by mapping $x \mapsto a_{21}$, $y \mapsto a_{12}$. It follows by construction that $\alpha(\varphi)$ is the equivalence class of ρ_A . Thus the map is surjective. \square

Proposition 5.2. *The functor h^{ver} is a versal hull of D^{ver} , R^{ver} is a versal deformation ring of $\chi_1 \oplus \chi_2$.*

Proof. According to [20, §2] we have to show that α induces a bijection $h^{\text{ver}}(k[\varepsilon]) \xrightarrow{\cong} D^{\text{ver}}(k[\varepsilon])$, and for every surjection $B \twoheadrightarrow A$ in \mathfrak{A} the map

$$h^{\text{ver}}(B) \rightarrow h^{\text{ver}}(A) \times_{D^{\text{ver}}(A)} D^{\text{ver}}(B)$$

is surjective. Both claims follow immediately from Lemma 5.1. \square

Remark 5.3. If $p = 2$ then it follows from the description of R^{ps} in the proof of Proposition 3.8 that $R^{\text{ver}} \cong \Lambda[[z_1, \dots, z_5, x, y]]/(z_5^2 + 2z_5 - xy)$. Thus R^{ver} has two irreducible components corresponding to the irreducible components of Λ . The universal deformation ring $R_{\chi_1\chi_2}$ of 1-dimensional representation $\chi_1\chi_2$ is isomorphic to $\Lambda[[x_1, x_2]]$. The map $R_{\chi_1\chi_2} \rightarrow R^{\text{ver}}$ induced by taking determinants is a map of Λ -algebras, and hence induces a bijection between their irreducible components. This verifies a conjecture of Böckle and Juschka in this case, see [4].

6. POTENTIALLY SEMI-STABLE DEFORMATION RINGS

Let R be either R_1 , R_2 , or R^{ver} , let ρ be either ρ_1 , ρ_2 , or $\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ and let ρ^{univ} be ρ_1^{univ} , ρ_2^{univ} or ρ^{ver} , respectively. If \mathfrak{p} is a maximal ideal of $R[1/p]$, then its residue field $\kappa(\mathfrak{p})$ is a finite extension of L . Let E be a finite extension of L , with the ring of integers \mathcal{O}_E and uniformizer ϖ_E . If $x : R \rightarrow E$ is a map of \mathcal{O} -algebras, then $\rho_x^{\text{univ}} := \rho^{\text{univ}} \otimes_{R,x} E$ is a continuous representation $\rho_x^{\text{univ}} : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \text{GL}_2(E)$. The image of $\mathcal{G}_{\mathbb{Q}_p}$ is contained in $\text{GL}_2(\mathcal{O}_E)$, and reducing this representation modulo ϖ_E we obtain ρ .

We say that x is potentially semi-stable if ρ_x^{univ} is a potentially semi-stable representation. In this case, to ρ_x^{univ} we can associate a pair of integers $\mathbf{w} = (a, b)$ with $a \leq b$, the Hodge–Tate weights, and a Weil–Deligne representation $\text{WD}(\rho_x^{\text{univ}})$. We fix $\mathbf{w} = (a, b)$ with $a < b$ and an a representation $\tau : I_{\mathbb{Q}_p} \rightarrow \text{GL}_2(L)$ of the inertia subgroup with an open kernel. Kisin has shown in [10] that the locus of x such that the Hodge–Tate weights of ρ_x^{univ} are equal to \mathbf{w} and $\text{WD}(\rho_x^{\text{univ}})|_{I_{\mathbb{Q}_p}} \cong \tau$ is closed in $\text{m-Spec } R[1/p]$. We will call such points of the p -adic Hodge type (\mathbf{w}, τ) . We let $\text{Spec } R(\mathbf{w}, \tau)$ be the closure of these points in $\text{Spec } R$ equipped with the reduced scheme structure. Thus $R(\mathbf{w}, \tau)$ is a reduced \mathcal{O} -torsion free quotient of R , characterized by the property that $x \in \text{m-Spec } R[1/p]$ lies in $\text{m-Spec } R(\mathbf{w}, \tau)[1/p]$ if and only if ρ_x^{univ} is of p -adic Hodge type (\mathbf{w}, τ) .

Remark 6.1. There are following variants of the set up above to which our results proved below apply, but we do not state them explicitly: one may consider potentially crystalline instead of potentially semi-stable points. In this case we will denote the corresponding ring by $R^{\text{cr}}(\mathbf{w}, \tau)$. One may fix a continuous character $\psi : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathcal{O}^\times$, and require that the representations have determinant equal to $\psi\varepsilon$, where ε is the cyclotomic character. In this case, we will denote the rings by R^ψ and $R^\psi(\mathbf{w}, \tau)$. Note that a necessary condition for $R^\psi(\mathbf{w}, \tau)$ to be non-zero is that

$\psi|_{I_{\mathbb{Q}_p}} = \varepsilon^{a+b-1} \det \tau$ and $\psi\varepsilon \equiv \chi_1\chi_2 \pmod{\varpi}$. One could also look at potentially crystalline representations with the fixed determinant.

Let $x : R \rightarrow E$ be a map of \mathcal{O} -algebras. It follows from Proposition 3.8 that the representation ρ_x^{univ} is reducible if and only if the reducibility ideal is mapped to zero under the composition $R^{\text{ps}} \rightarrow R \xrightarrow{x} E$. This implies that if ρ_x^{univ} is irreducible, then it remains irreducible after extending scalars. Let us assume that x is potentially semi-stable of p -adic Hodge type (\mathbf{w}, τ) . If ρ_x^{univ} is reducible then it is an extension $0 \rightarrow \delta_1 \rightarrow \rho_x^{\text{univ}} \rightarrow \delta_2 \rightarrow 0$, where $\delta_1, \delta_2 : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathcal{O}_E^\times$ are continuous characters. Since ρ_x^{univ} is potentially semi-stable, both δ_1 and δ_2 are potentially semi-stable. Moreover, we may assume that the Hodge-Tate weight of $\delta_1\delta_2^{-1}$ is at least 1, this holds automatically if the extension is non-split. Following Hu–Tan [19] we say that x is of reducibility type 1 if $\delta_1 \equiv \chi_1 \pmod{\varpi_E}$ (equivalently $\delta_2 \equiv \chi_2 \pmod{\varpi_E}$). We say that x is of reducibility type 2 if $\delta_1 \equiv \chi_2 \pmod{\varpi_E}$. We say that x is of reducibility type irr, if ρ_x^{univ} is irreducible.

Let $*$ be one of the indices 1, 2 or irr, we define I_*^{ver} to be the ideal of R^{ver} and I_*^{ps} to be the ideal of R^{ps} given by

$$I_*^{\text{ver}} := R^{\text{ver}} \cap \bigcap_x \mathfrak{m}_x, \quad I_*^{\text{ps}} := R^{\text{ps}} \cap I_*^{\text{ver}},$$

where the intersection is taken over all $x \in \text{m-Spec } R^{\text{ver}}(\mathbf{w}, \tau)[1/p]$ of reducibility type $*$.

Lemma 6.2. *Let A be a local noetherian ring. Let $B = A[[x, y]]/(xy - c)$, with $c \in \mathfrak{m}_A$. Then B is A -flat and $\dim B = \dim A + 1$. If A is reduced then B is reduced.*

Proof. Let $z = x + y$ then $B = A[[z]][x]/(x^2 - zx + c)$. Thus B is a free $A[[z]]$ -module of rank 2. This implies the claims about flatness and dimension. If \mathfrak{p} is a prime of A then $\mathfrak{q} := \mathfrak{p}A[[z]]$ is a prime of $A[[z]]$. Since z is transcendental over $\kappa(\mathfrak{p})$, $z^2 - 4c$ cannot be zero in $\kappa(\mathfrak{q})$. Thus $\kappa(\mathfrak{q})[x]/(x^2 - zx + c)$ is reduced, and the subring $A/\mathfrak{p}[[z]][x]/(x^2 - zx + c)$ is also reduced. If A is reduced then we may embed A into $\prod_{\mathfrak{p}} A/\mathfrak{p}$, where the product is taken over all the minimal primes of A . Hence, B can be embedded into the product of reduced rings $\prod_{\mathfrak{p}} A/\mathfrak{p}[[z]][x]/(x^2 - zx + c)$. \square

Lemma 6.3. *Let A be a local noetherian ring and let $\mathfrak{p} \in \text{Spec } A$ be such that $\dim A/\mathfrak{p} = \dim A$. Let $B = A[[x, y]]/(xy - c)$, with $c \in \mathfrak{p}$. Let \mathfrak{q} be the ideal of B generated by \mathfrak{p} and x . Then \mathfrak{q} is a prime ideal with $\dim B = \dim B/\mathfrak{q} = \dim A + 1$. Moreover, $e(A/\mathfrak{p}) = e(B/\mathfrak{q})$, $\ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = \ell_{B_{\mathfrak{q}}}(B_{\mathfrak{q}})$*

Proof. Since $B/\mathfrak{q} \cong (A/\mathfrak{p})[[y]]$ we get that \mathfrak{q} is a prime ideal of B , $e(A/\mathfrak{p}) = e(B/\mathfrak{q})$ and $\dim B/\mathfrak{q} = \dim A/\mathfrak{p} + 1 = \dim A + 1 = \dim B$, where the last equality follows from Lemma 6.2. Since B is A -flat by Lemma 6.2, $B_{\mathfrak{q}}$ is $A_{\mathfrak{p}}$ -flat. Since $\mathfrak{p}B_{\mathfrak{q}}$ is the maximal ideal of $B_{\mathfrak{q}}$, flatness implies that $\ell(A_{\mathfrak{p}}) = \ell(B_{\mathfrak{q}})$. \square

Lemma 6.4. *The map (12) induces isomorphisms:*

$$R^{\text{ps}}/I_1^{\text{ps}}[[y]] \cong R^{\text{ver}}/I_1^{\text{ver}}, \quad R^{\text{ps}}/I_2^{\text{ps}}[[x]] \cong R^{\text{ver}}/I_2^{\text{ver}},$$

$$R^{\text{ps}}/I_{\text{irr}}^{\text{ps}}[[x, y]]/(xy - c) \cong R^{\text{ver}}/I_{\text{irr}}^{\text{ver}}.$$

Proof. Let $*$ be one of the following indices: 1, 2 or irr. Then (12) induces a surjection:

$$(14) \quad R^{\text{ps}}/I_*^{\text{ps}}[[x, y]]/(xy - c) \cong R^{\text{ver}}/I_*^{\text{ps}}R^{\text{ver}} \rightarrow R^{\text{ver}}/I_*^{\text{ver}}.$$

We will deal with the irreducible case first. To ease the notation let $A = R^{\text{ps}}/I_{\text{irr}}^{\text{ps}}$ and let $B = R^{\text{ver}}/I_{\text{irr}}^{\text{ps}}R^{\text{ver}}$. Lemma 6.2 implies that B is reduced and A -flat. Since by construction a subset of $\text{Spec } A[1/p]$ is dense in $\text{Spec } A$, A is \mathcal{O} -torsion free. Flatness implies that B is \mathcal{O} -torsion free, hence $\text{Spec } B[1/p]$ is dense in $\text{Spec } B$. Since $B[1/p]$ is Jacobson, $\text{m-Spec } B[1/p]$ is dense in $\text{Spec } B$. The reducible locus in $\text{Spec } B$ is given by $c = 0$, and is isomorphic to $\text{Spec}(A/c)[[x, y]]/(xy)$. Now $\dim A/c < \dim A$, since otherwise points of type irr would have to be dense in $\text{Spec } A/c$. Hence the reducible locus in $\text{Spec } B$ has codimension 1. Thus the subset Σ' of $\text{m-Spec } B[1/p]$, consisting of those maximal ideals, which correspond to absolutely irreducible representations, is dense in $\text{Spec } B$. Since an absolutely irreducible representation is determined up to isomorphism by its trace, Σ' lies in the image of $\text{Spec } R^{\text{ver}}/I_{\text{irr}}^{\text{ver}} \rightarrow \text{Spec } B$. Since this map is a closed immersion, it is a homeomorphism. Since B is reduced, we obtain the assertion.

If $*$ = 1 then c is contained in every maximal ideal of type 1, and hence $c \in I_1^{\text{ps}}$. It follows from the construction of the versal representation that any maximal ideal of $R^{\text{ver}}(\mathbf{w}, \tau)[1/p]$ of type 1 will contain y , and any maximal ideal of $R^{\text{ver}}/(I_1^{\text{ps}}, x)[1/p]$ is of type 1. Thus (14) induces a closed immersion

$$\text{Spec } R^{\text{ver}}/I_1^{\text{ver}} \hookrightarrow \text{Spec } R^{\text{ver}}/(I_1^{\text{ps}}, x),$$

and $\text{m-Spec } R^{\text{ver}}/(I_1^{\text{ps}}, x)[1/p]$ lies in its image. Now $R^{\text{ver}}/(I_1^{\text{ps}}, x) \cong R^{\text{ps}}/I_1^{\text{ps}}[[y]]$, and hence is reduced and \mathcal{O} -torsion free. The same argument as in the irreducible case allows to conclude. If $*$ = 2 then the argument is the same interchanging x and y . \square

Lemma 6.5. *The isomorphisms $t_1 : R^{\text{ps}} \xrightarrow{\cong} R_1$, $t_2 : R^{\text{ps}} \xrightarrow{\cong} R_2$ induces isomorphisms: $R^{\text{ps}}/I_1^{\text{ps}} \cap I_{\text{irr}}^{\text{ps}} \cong R_1(\mathbf{w}, \tau)$, $R^{\text{ps}}/I_2^{\text{ps}} \cap I_{\text{irr}}^{\text{ps}} \cong R_2(\mathbf{w}, \tau)$.*

Proof. Since the rings are \mathcal{O} -torsion free and reduced, it is enough to show that the maps induce a bijection on maximal spectra after inverting p . We will show the statement for $R_1(\mathbf{w}, \tau)$, the proof for $R_2(\mathbf{w}, \tau)$ is the same. Since t_1 is an isomorphism it induces a bijection between $\text{m-Spec } R_1[1/p]$ and $\text{m-Spec } R^{\text{ps}}[1/p]$ and hence it is enough to show that every $x \in \text{m-Spec } R_1(\mathbf{w}, \tau)[1/p]$ is mapped to $V(I_1^{\text{ps}} \cap I_{\text{irr}}^{\text{ps}})$ and every $y \in V(I_1^{\text{ps}} \cap I_{\text{irr}}^{\text{ps}}) \cap \text{m-Spec } R^{\text{ps}}[1/p]$ has a preimage in $\text{m-Spec } R_1(\mathbf{w}, \tau)[1/p]$.

Let E be a finite extension of L with the ring of integers \mathcal{O}_E and let $x : R_1(\mathbf{w}, \tau) \rightarrow E$ be an E -valued point of $\text{Spec } R_1(\mathbf{w}, \tau)$. Let $\rho_x := \rho_1^{\text{univ}} \otimes_{R_1, x} E$. The image of R_1 under x is contained in \mathcal{O}_E , and we let $\rho_x^0 := \rho_1^{\text{univ}} \otimes_{R_1, x} \mathcal{O}_E$. Then ρ_x^0 is a $\mathcal{G}_{\mathbb{Q}_p}$ -invariant \mathcal{O}_E -lattice in ρ_x , and its reduction modulo the uniformizer ϖ_E , is isomorphic to ρ_1 .

If ρ_x is reducible then it is an extension $0 \rightarrow \delta_1 \rightarrow \rho_x \rightarrow \delta_2 \rightarrow 0$, where $\delta_1, \delta_2 : \mathcal{G}_{\mathbb{Q}_p} \rightarrow E^\times$ are continuous characters. This extension is non-split, as the reduction of ρ_x^0 modulo ϖ_E is a non-split extension of distinct characters. Moreover, δ_1 is congruent to χ_1 and δ_2 is congruent to χ_2 modulo ϖ_E . Since $x \in \text{Spec } R_1(\mathbf{w}, \tau)$, ρ_x is potentially semi-stable, hence both δ_1 and δ_2 are potentially semi-stable, and the Hodge-Tate weight of δ_1 is greater than the Hodge-Tate weight of δ_2 .

By conjugating ρ_x^0 with $\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}$, for a suitable $n \in \mathbb{Z}$, we will obtain a new $\mathcal{G}_{\mathbb{Q}_p}$ -invariant \mathcal{O}_E -lattice in ρ_x , such that its reduction modulo ϖ_E is congruent to $\chi_1 \oplus \chi_2$. This gives an \mathcal{O}_E -valued point in $\text{Spec } R^{\text{ver}}(\mathbf{w}, \tau)$, which has the same trace as ρ_x . Hence, the map $\text{Spec } R_1(\mathbf{w}, \tau) \rightarrow \text{Spec } R^{\text{ps}}$ maps x into $V(I_1^{\text{ps}})$.

Conversely, let y be an E -valued point of $R^{\text{ps}}/I_1^{\text{ps}}$, then the determinant corresponding to y is a pair $(\delta_1 + \delta_2, \delta_1 \delta_2)$, such that there is $z : R^{\text{ver}}(\mathbf{w}, \tau) \rightarrow E$ fitting into the exact sequence $0 \rightarrow \delta_1 \rightarrow \rho^{\text{ver}} \otimes_{R^{\text{ver}}, z} E \rightarrow \delta_2 \rightarrow 0$, such that $\delta_1 \equiv \chi_1 \pmod{\varpi_E}$, $\delta_2 \equiv \chi_2 \pmod{\varpi_E}$ and the Hodge–Tate weight of $\delta_1 \delta_2^{-1}$ is at least 1. Since $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\delta_2, \delta_1)$ is non-zero, there is a non-split extension $0 \rightarrow \delta_1 \rightarrow \tilde{\rho} \rightarrow \delta_2 \rightarrow 0$. Since the Hodge–Tate weight of $\delta_1 \delta_2^{-1}$ is at least 1, the representation $\tilde{\rho}$ is potentially semi-stable of p -adic Hodge type (\mathbf{w}, τ) . Since $\chi_1 \chi_2^{-1} \neq 1, \omega^{\pm 1}$, $\delta_1 \delta_2^{-1} \neq 1, \varepsilon^{\pm 1}$, the extension is in fact potentially crystalline.

We may choose a $\mathcal{G}_{\mathbb{Q}_p}$ -invariant \mathcal{O}_E -lattice ρ^0 in ρ , such that $\rho^0 \otimes_{\mathcal{O}_E} k$ is a non-split extension of χ_1 by χ_2 . Since $\text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\chi_2, \chi_1)$ is one dimensional, this representation is isomorphic to ρ_1 , and thus $\tilde{\rho}$ gives us an E -valued point of $\text{Spec } R_1(\mathbf{w}, \tau)$. Hence, y lies in the image of $\text{Spec } R_1(\mathbf{w}, \tau) \rightarrow \text{Spec } R^{\text{ps}}$.

In the irreducible case the argument is easier. If ρ_x is irreducible then after extending scalars to $E' := E[\sqrt{\varpi_E}]$, we will be able to find a $\mathcal{G}_{\mathbb{Q}_p}$ -invariant $\mathcal{O}_{E'}$ lattice in $\rho_x \otimes_E E'$ with reduction modulo $\varpi_{E'}$ isomorphic to $\chi_1 \oplus \chi_2$ by arguing in the same way as in the reducible case. This gives us an E' -valued point in $\text{Spec } R^{\text{ver}}(\mathbf{w}, \tau)$. As remarked after Remark 6.1 ρ_x is absolutely irreducible. Since the representation obtained by extending scalars is irreducible and has the same trace as ρ_x , we deduce that the image of x in $\text{m-Spec } R^{\text{ps}}[1/p]$ lies in $V(I_{\text{irr}}^{\text{ps}})$.

Conversely, let y be an E -valued point of $\text{Spec } R^{\text{ps}}/I_{\text{irr}}^{\text{ps}}$. Let x be the image of y in $\text{m-Spec } R_1[1/p]$ under the map induced by the isomorphism $t_1 : R^{\text{ps}} \xrightarrow{\cong} R_1$. It follows from the definition of $I_{\text{irr}}^{\text{ps}}$ that there is a finite extension E' of E and an E' -valued point z of $R^{\text{ver}}(\mathbf{w}, \tau)$, such that ρ_z^{ver} is irreducible with trace equal to t_y^{univ} . In particular, ρ_z^{ver} and ρ_x have the same trace. As remarked after Remark 6.1 ρ_z^{ver} is absolutely irreducible, thus the equality of traces implies that ρ_z^{ver} is isomorphic to $\rho_x \otimes_E E'$ as $\mathcal{G}_{\mathbb{Q}_p}$ -representations. Hence, ρ_x is potentially semi-stable of p -adic Hodge type equal to (\mathbf{w}, τ) . This implies that x lies in $\text{m-Spec } R_1(\mathbf{w}, \tau)[1/p]$. \square

Recall, [21, §V.A], that the group of d -dimensional cycles $\mathcal{Z}_d(A)$ of a noetherian ring A is a free abelian group generated by $\mathfrak{p} \in \text{Spec } A$ with $\dim A/\mathfrak{p} = d$.

Lemma 6.6. *Let $d = \dim R^{\text{ver}}(\mathbf{w}, \tau)$ then there is an equality of cycles in $\mathcal{Z}_d(R^{\text{ver}})$:*

$$z_d(R^{\text{ver}}(\mathbf{w}, \tau)) = z_d(R^{\text{ver}}/I_1^{\text{ver}} \oplus R^{\text{ver}}/I_{\text{irr}}^{\text{ver}} \oplus R^{\text{ver}}/I_2^{\text{ver}}).$$

Proof. Since $R^{\text{ver}}(\mathbf{w}, \tau) = R^{\text{ver}}/(I_1^{\text{ver}} \cap I_{\text{irr}}^{\text{ver}} \cap I_2^{\text{ver}})$, it is enough to show that $\mathfrak{q} \in \text{Spec } R^{\text{ver}}$ with $\dim R^{\text{ver}}/\mathfrak{q} = d$ can lie in the support of at most one of the modules $R^{\text{ver}}/I_1^{\text{ver}}$, $R^{\text{ver}}/I_{\text{irr}}^{\text{ver}}$, $R^{\text{ver}}/I_2^{\text{ver}}$. If \mathfrak{q} lies in the support of $R^{\text{ver}}/I_{\text{irr}}^{\text{ver}}$, then for dimension reasons it has to be a minimal prime in $V(I_{\text{irr}}^{\text{ver}})$, and thus points of type $*$ are dense in $V(\mathfrak{q})$. If $*$ is 1 or 2 then $c \in \mathfrak{q}$ and hence $V(\mathfrak{q})$ does not contain points of type irr. If \mathfrak{q} lies in the support of both $R^{\text{ver}}/I_1^{\text{ver}}$ and $R^{\text{ver}}/I_2^{\text{ver}}$ then it will lie in $V((x, y))$, which has codimension 1, as follows from Lemma 6.4. \square

7. THE BREUIL–MÉZARD CONJECTURE

Recall that the reducible locus in R_1 , R_2 , R^{ps} and R^{ver} is defined by the equation $c = 0$. The isomorphism $t_1 : R^{\text{ps}} \xrightarrow{\cong} R_1$, $t_2 : R^{\text{ps}} \xrightarrow{\cong} R_2$, (12) induce isomorphisms:

$$t_1 : R^{\text{ps}}/c \xrightarrow{\cong} R_1/c, \quad t_2 : R^{\text{ps}}/c \xrightarrow{\cong} R_2/c, \quad R^{\text{ps}}/c[[x, y]]/(xy) \xrightarrow{\cong} R^{\text{ver}}/c.$$

For $\mathfrak{p}_1 \in \text{Spec } R_1/c$ let \mathfrak{q}_1 be the ideal of R^{ver}/c defined by $\mathfrak{q}_1 := (t_1^{-1}(\mathfrak{p}_1), x)$. Then $R^{\text{ver}}/\mathfrak{q}_1 \cong R_1/\mathfrak{p}_1[[y]]$, and so $\mathfrak{q}_1 \in \text{Spec } R^{\text{ver}}/c$, $\dim R^{\text{ver}}/\mathfrak{q}_1 = \dim R_1/\mathfrak{p}_1 + 1$, $e(R^{\text{ver}}/\mathfrak{q}_1) = e(R_1/\mathfrak{p}_1)$. Similarly, for $\mathfrak{p}_2 \in \text{Spec } R_2/c$ we let $\mathfrak{q}_2 := (t_2^{-1}(\mathfrak{p}_2), y)$, then $R^{\text{ver}}/\mathfrak{q}_2 \cong R_2/\mathfrak{p}_2[[x]]$, $\dim R^{\text{ver}}/\mathfrak{q}_2 = \dim R_2/\mathfrak{p}_2 + 1$, $e(R^{\text{ver}}/\mathfrak{q}_2) = e(R_2/\mathfrak{p}_2)$. Hence, for all $0 \leq i \leq \dim R^{\text{ps}}/(c)$ the map $\mathfrak{p}_1 \mapsto \mathfrak{q}_1$, $\mathfrak{p}_2 \mapsto \mathfrak{q}_2$ induces an injection

$$(15) \quad \alpha : \mathcal{Z}_i(R_1/(c)) \oplus \mathcal{Z}_i(R_2/(c)) \hookrightarrow \mathcal{Z}_{i+1}(R^{\text{ver}}/(c)).$$

Moreover, this map preserves Hilbert–Samuel multiplicities.

Theorem 7.1. *Assume that $\text{Spec } R_1(\mathbf{w}, \tau)/\varpi$ (equivalently, $\text{Spec } R_2(\mathbf{w}, \tau)/\varpi$) is contained in the reducible locus. Let d be the dimension of $R^{\text{ver}}(\mathbf{w}, \tau)$ then there is an equality of $(d-1)$ -dimensional cycles:*

$$z_{d-1}(R^{\text{ver}}(\mathbf{w}, \tau)/\varpi) = \alpha(z_{d-2}(R_1(\mathbf{w}, \tau)/\varpi) + z_{d-2}(R_2(\mathbf{w}, \tau)/\varpi)).$$

Proof. If M_1 and M_2 are finally generated d -dimensional modules over a noetherian ring R , such that $z_d(M_1) = z_d(M_2)$, and $x \in R$ is both M_1 - and M_2 -regular, then $z_{d-1}(M_1/x) = z_{d-1}(M_2/x)$, [8, 2.2.13]. This fact and Lemma 6.6 imply that

$$(16) \quad \begin{aligned} z_{d-1}(R^{\text{ver}}(\mathbf{w}, \tau)/\varpi) &= z_{d-1}(R^{\text{ver}}/(I_1^{\text{ver}}, \varpi)) + z_{d-1}(R^{\text{ver}}/(I_{\text{irr}}^{\text{ver}}, \varpi)) \\ &\quad + z_{d-1}(R^{\text{ver}}/(I_2^{\text{ver}}, \varpi)). \end{aligned}$$

Similarly from Lemma 6.5 one obtains

$$(17) \quad z_{d-2}(R_1(\mathbf{w}, \tau)/\varpi) = z_{d-2}(R_1/(t_1(I_1^{\text{ps}}), \varpi)) + z_{d-2}(R_1/(t_1(I_{\text{irr}}^{\text{ps}}), \varpi)).$$

$$(18) \quad z_{d-2}(R_2(\mathbf{w}, \tau)/\varpi) = z_{d-2}(R_2/(t_2(I_2^{\text{ps}}), \varpi)) + z_{d-2}(R_2/(t_2(I_{\text{irr}}^{\text{ps}}), \varpi)).$$

It is immediate from Lemma 6.4 and the definition of α that

$$\begin{aligned} \alpha(z_{d-2}(R_1/(t_1(I_1^{\text{ps}}), \varpi))) &= z_{d-1}(R^{\text{ver}}/(I_1^{\text{ver}}, \varpi)), \\ \alpha(z_{d-2}(R_2/(t_1(I_1^{\text{ps}}), \varpi))) &= z_{d-1}(R^{\text{ver}}/(I_2^{\text{ver}}, \varpi)). \end{aligned}$$

The assumption that the special fibre of the potentially semi-stable ring is contained in the reducible locus implies that c is nilpotent in $R^{\text{ps}}/(I_{\text{irr}}^{\text{ps}}, \varpi)$. It follows from Lemmas 6.4, 6.3 that

$$z_{d-1}(R^{\text{ver}}/(I_{\text{irr}}^{\text{ver}}, \varpi)) = \alpha(z_{d-2}(R_1/(t_1(I_{\text{irr}}^{\text{ps}}), \varpi)) + z_{d-2}(R_2/(t_2(I_{\text{irr}}^{\text{ps}}), \varpi))).$$

□

Corollary 7.2. *Under the assumption of Theorem 7.1, we have an equality of the Hilbert–Samuel multiplicities:*

$$e(R^{\text{ver}}(\mathbf{w}, \tau)/\varpi) = e(R_1(\mathbf{w}, \tau)/\varpi) + e(R_2(\mathbf{w}, \tau)/\varpi).$$

Proof. This follows from the fact that (15) preserves Hilbert–Samuel multiplicities and Theorem 7.1. □

In [18], Henniart has shown the existence of a smooth irreducible representation $\sigma(\tau)$ (resp. $\sigma^{\text{cr}}(\tau)$) of $K := \text{GL}_2(\mathbb{Z}_p)$ on an L -vector space, such that if π is a smooth absolutely irreducible infinite dimensional representation of $G := \text{GL}_2(\mathbb{Q}_p)$ and $\text{LL}(\pi)$ is the Weil-Deligne representation attached to π by the classical local Langlands correspondence then $\text{Hom}_K(\sigma(\tau), \pi) \neq 0$ (resp. $\text{Hom}_K(\sigma^{\text{cr}}(\tau), \pi) \neq 0$) if and only if $\text{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ (resp. $\text{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ and the monodromy operator $N = 0$). We have $\sigma(\tau) \cong \sigma^{\text{cr}}(\tau)$ in all cases, except if $\tau \cong \chi \oplus \chi$, then $\sigma(\tau) \cong \tilde{\text{st}} \otimes \chi \circ \det$ and $\sigma^{\text{cr}}(\tau) \cong \chi \circ \det$, where $\tilde{\text{st}}$ is the Steinberg representation of $\text{GL}_2(\mathbb{F}_p)$, and we view χ as a character of \mathbb{Z}_p^\times via the local class field theory.

We let $\sigma(\mathbf{w}, \tau) := \sigma(\tau) \otimes \text{Sym}^{b-a-1} L^2 \otimes \det^a$. Then $\sigma(\mathbf{w}, \tau)$ is a finite dimensional L -vector space. Since K is compact and the action of K on $\sigma(\mathbf{w}, \tau)$ is continuous, there is a K -invariant \mathcal{O} -lattice Θ in $\sigma(\mathbf{w}, \tau)$. Then $\Theta/(\varpi)$ is a smooth finite length k -representation of K , and we let $\overline{\sigma(\mathbf{w}, \tau)}$ be its semi-simplification. One may show that $\overline{\sigma(\mathbf{w}, \tau)}$ does not depend on the choice of a lattice. For each smooth irreducible k -representation σ of K we let $m_\sigma(\mathbf{w}, \tau)$ be the multiplicity with which σ occurs in $\overline{\sigma(\mathbf{w}, \tau)}$. We let $\sigma^{\text{cr}}(\mathbf{w}, \tau) := \sigma^{\text{cr}}(\tau) \otimes \text{Sym}^{b-a-1} L^2 \otimes \det^a$ and let $m_\sigma^{\text{cr}}(\mathbf{w}, \tau)$ be the multiplicity of σ in $\overline{\sigma^{\text{cr}}(\mathbf{w}, \tau)}$.

Theorem 7.3. *If the determinant is fixed let $d = 3$, otherwise let $d = 4$. Assume that there are finite sets $\{\mathcal{C}_{1,\sigma}\}_\sigma \subset \mathcal{Z}_{d-2}(R_1/\varpi)$, $\{\mathcal{C}_{2,\sigma}\}_\sigma \subset \mathcal{Z}_{d-2}(R_2/\varpi)$ such that for all p -adic Hodge types (\mathbf{w}, τ) we have equalities*

$$\begin{aligned} z_{d-2}(R_1(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma m_\sigma(\mathbf{w}, \tau) \mathcal{C}_{1,\sigma}, & z_{d-2}(R_2(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma m_\sigma(\mathbf{w}, \tau) \mathcal{C}_{2,\sigma}. \\ z_{d-2}(R_1^{\text{cr}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma m_\sigma^{\text{cr}}(\mathbf{w}, \tau) \mathcal{C}_{1,\sigma}, & z_{d-2}(R_2^{\text{cr}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma m_\sigma^{\text{cr}}(\mathbf{w}, \tau) \mathcal{C}_{2,\sigma}. \end{aligned}$$

Then $\text{Spec } R_1(\mathbf{w}, \tau)/\varpi$, $\text{Spec } R_2(\mathbf{w}, \tau)/\varpi$ are contained in the reducible locus of $\text{Spec } R_1$ and $\text{Spec } R_2$ respectively and for all p -adic Hodge types (\mathbf{w}, τ) we have

$$\begin{aligned} z_{d-1}(R^{\text{ver}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma (m_\sigma(\mathbf{w}, \tau) \alpha(\mathcal{C}_{1,\sigma}) + m_{2,\sigma}(\mathbf{w}, \tau) \alpha(\mathcal{C}_{2,\sigma})), \\ z_{d-1}(R^{\text{ver,cr}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma (m_\sigma^{\text{cr}}(\mathbf{w}, \tau) \alpha(\mathcal{C}_{1,\sigma}) + m_\sigma^{\text{cr}}(\mathbf{w}, \tau) \alpha(\mathcal{C}_{2,\sigma})). \end{aligned}$$

In particular,

$$\begin{aligned} e(R^{\text{ver}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma (m_\sigma(\mathbf{w}, \tau) e(\mathcal{C}_{1,\sigma}) + m_\sigma(\mathbf{w}, \tau) e(\mathcal{C}_{2,\sigma})), \\ e(R^{\text{ver,cr}}(\mathbf{w}, \tau)/\varpi) &= \sum_\sigma (m_\sigma^{\text{cr}}(\mathbf{w}, \tau) e(\mathcal{C}_{1,\sigma}) + m_\sigma^{\text{cr}}(\mathbf{w}, \tau) e(\mathcal{C}_{2,\sigma})). \end{aligned}$$

Proof. Each σ is isomorphic to a representation of the form $\text{Sym}^r k^2 \otimes \det^s$ with $0 \leq r \leq p-1$, and $0 \leq s \leq p-2$. The pair (r, s) is uniquely determined by σ . Let $\mathbf{w}(\sigma) := (s, s+r+1)$. Then $\sigma^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1}) = \text{Sym}^r L^2 \otimes \det^s$, and $\overline{\sigma^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1})} \cong \sigma$. Thus $m_\sigma^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1}) = 1$, and $m_{\sigma'}^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1}) = 0$ for all $\sigma' \not\cong \sigma$. Hence, the assumption implies that for all σ :

$$\mathcal{C}_{1,\sigma} = z_{d-2}(R_1^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1})), \quad \mathcal{C}_{2,\sigma} = z_{d-2}(R_2^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1})).$$

Since $\text{Spec } R_1^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1})$ and $\text{Spec } R_2^{\text{cr}}(\mathbf{w}(\sigma), \mathbf{1} \oplus \mathbf{1})$ are contained in the reducible locus by [9, Lem.3.5], the assertion follows immediately from Theorem 7.1 and the fact that (15) preserves Hilbert–Samuel multiplicities. \square

Remark 7.4. Let R_1^\square , R_2^\square , R^\square be the framed deformation rings of ρ_1 , ρ_2 and ρ respectively, and let $R_1^\square(\mathbf{w}, \tau)$, $R_2^\square(\mathbf{w}, \tau)$ and $R^\square(\mathbf{w}, \tau)$ denote the quotients, which parameterize potentially semi-stable lifts of type (\mathbf{w}, τ) . It follows from [9, Prop. 2.1] that R_1^\square is formally smooth over R_1 of relative dimension 3, R_2^\square is formally smooth over R_2 of relative dimension 3, R^\square is formally smooth over R^{ver} of relative dimension 2. Since these framing variables only keep track of the chosen basis, we deduce that $R_1^\square(\mathbf{w}, \tau)$, $R_2^\square(\mathbf{w}, \tau)$ and $R^\square(\mathbf{w}, \tau)$ are formally smooth over $R_1(\mathbf{w}, \tau)$, $R_2(\mathbf{w}, \tau)$ and $R^{\text{ver}}(\mathbf{w}, \tau)$ of relative dimension 3, 3 and 2 respectively. This allows to use Theorem 7.3 to deduce an analogous statement for the framed deformations rings. Moreover, one may additionally consider potentially crystalline lifts and/or fix the determinant.

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